Introduction to the Gamma Function

Pascal Sebah and Xavier Gourdon
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Abstract

An elementary introduction to the celebrated gamma function $\Gamma(x)$ and its various representations. Some of its most important properties are described.

1 Introduction

The *gamma function* was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values. Later, because of its great importance, it was studied by other eminent mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), ... as well as many others.

The gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study.

It also appears in various area as asymptotic series, definite integration, hypergeometric series, Riemann zeta function, number theory ...

Some of the historical background is due to Godefroy's beautiful essay on this function [9] and the more modern textbook [3] is a complete study.

2 Definitions of the gamma function

2.1 Definite integral

During the years 1729 and 1730 ([9], [12]), Euler introduced an analytic function which has the property to interpolate the factorial whenever the argument of the function is an integer. In a letter from January 8, 1730 to Christian Goldbach he proposed the following definition :

Definition 1 *(Euler, 1730) Let* $x > 0$

$$\Gamma(x) = \int_0^1 (-\log(t))^{x-1}\, dt. \tag{1}$$

By elementary changes of variables this historical definition takes the more usual forms :

Theorem 2 *For $x > 0$*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \tag{2}$$

or sometimes

$$\Gamma(x) = 2 \int_0^\infty t^{2x-1} e^{-t^2} dt. \tag{3}$$

Proof. *Use respectively the changes of variable $u = -\log(t)$ and $u^2 = -\log(t)$ in (1).* ■

From this theorem, we see that the *gamma function* $\Gamma(x)$ (or the *Eulerian integral of the second kind*) is well defined and analytic for $x > 0$ (and more generally for complex numbers x with positive real part).

The notation $\Gamma(x)$ is due to Legendre in 1809 [11] while Gauss expressed it by $\Pi(x)$ (which represents $\Gamma(x+1)$).

The derivatives can be deduced by differentiating under the integral sign of (2)

$$\Gamma'(x) = \int_0^\infty t^{x-1} e^{-t} \log(t) dt,$$

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1} e^{-t} \log^n(t) dt.$$

2.1.1 Functional equation

We have obviously

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \tag{4}$$

and for $x > 0$, an integration by parts yields

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = [-t^x e^{-t}]_0^\infty + x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x), \tag{5}$$

and the relation $\Gamma(x+1) = x\Gamma(x)$ is the important *functional equation.*

For integer values the functional equation becomes

$$\Gamma(n+1) = n!,$$

and it's why the gamma function can be seen as an extension of the factorial function to real non null positive numbers.

A natural question is to determine if the gamma function is the only solution of the functional equation ? The answer is clearly no as may be seen if we consider, for example, the functions $\cos(2m\pi x)\Gamma(x)$, where m is any non null integer and which satisfy both (4) and (5). But the following result states that under an additional condition the gamma function is the only solution of this equation.

Theorem 3 *(Bohr-Mollerup, 1922, [6]) There is a* ***unique*** *function* $f :]0, +\infty[\to]0, +\infty[$ *such as* $\log(f(x))$ *is convex and*

$$f(1) = 1,$$
$$f(x+1) = xf(x).$$

Proof. *An elementary one is given in [2].* ■

Other conditions may also work as well, see again [2].

It's also possible to extend this function to negative values by inverting the functional equation (which becomes a definition identity for $-1 < x < 0$)

$$\Gamma(x) = \frac{\Gamma(x+1)}{x},$$

and for example $\Gamma(-1/2) = -2\Gamma(1/2)$. Reiteration of this identity allows to define the gamma function on the whole real axis except on the negative integers $(0, -1, -2, ...)$. For any non null integer n, we have

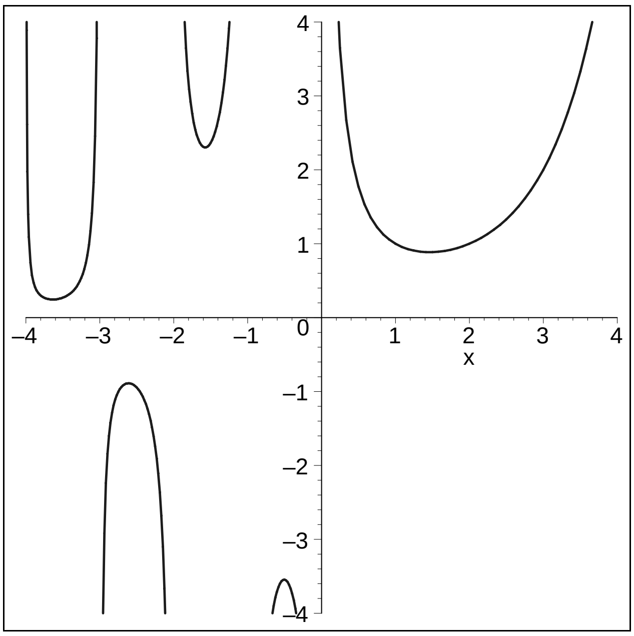
$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)...(x+n-1)} \qquad x+n > 0. \tag{6}$$

Suppose that $x = -n + h$ with h being small, then

$$\Gamma(x) = \frac{\Gamma(1+h)}{h(h-1)...(h-n)} \sim \frac{(-1)^n}{n!h} \qquad \text{when } h \to 0,$$

so $\Gamma(x)$ possesses simple poles at the negative integers $-n$ with residue $(-1)^n/n!$ (see the plot of the function 2.1.1).

In fact, also by mean of relation (6), the gamma function can be defined in the whole complex plane.



2.2 Another definition by Euler and Gauss

In another letter written in October 13, 1729 also to his friend Goldbach, Euler gave another equivalent definition for $\Gamma(x)$.

Definition 4 *(Euler, 1729 and Gauss, 1811) Let $x > 0$ and define*

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)...(x+p)} = \frac{p^x}{x(1+x/1)...(1+x/p)}, \tag{7}$$

then

$$\Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \tag{8}$$

(Check the existence of this limit). This approach, using an infinite product, was also chosen, in 1811, by Gauss in his study of the gamma function [8].

Clearly

$$\Gamma_p(1) = \frac{p!}{1(1+1)...(1+p)}p = \frac{p}{p+1}, \text{ and}$$
$$\Gamma_p(x+1) = \frac{p!p^{x+1}}{(x+1)...(x+p+1)} = \frac{p}{x+p+1}x\Gamma_p(x),$$

hence

$$\Gamma(1) = 1,$$
$$\Gamma(x+1) = x\Gamma(x).$$

We retrieve the functional equation verified by $\Gamma(x)$.

It's interesting to observe that the definition is still valid for negative values of x, except on the poles $(0, -1, -2, ...)$. Using this formulation is often more convenient to establish new properties of the gamma function.

2.3 Weierstrass formula

The relation

$$p^x = e^{x\log(p)} = e^{x(\log(p)-1-1/2-...-1/p)}e^{x+x/2+...+x/p},$$

entails

$$\Gamma_p(x) = \frac{1}{x}\frac{1}{x+1}\frac{2}{x+2}...\frac{p}{x+p}p^x = \frac{e^{x(\log(p)-1-1/2-...-1/p)}e^{x+x/2+...+x/p}}{x(1+x)(1+x/2)...(1+x/p)},$$
$$\Gamma_p(x) = e^{x(\log(p)-1-1/2-...-1/p)}\frac{1}{x}\frac{e^x}{1+x}\frac{e^{x/2}}{1+x/2}\cdots\frac{e^{x/p}}{1+x/p}.$$

Now *Euler's constant* is defined by

$$\gamma = \lim_{p \to \infty} \left(1 + \frac{1}{2} + ... + \frac{1}{p} - \log(p)\right) = 0.5772156649015328606...,$$

and therefore follows the *Weierstrass form* of the gamma function.

Theorem 5 *(Weierstrass) For any real number x, except on the negative integers $(0, -1, -2, ...)$, we have the infinite product*

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{p=1}^{\infty} \left(1 + \frac{x}{p}\right) e^{-x/p}. \tag{9}$$

From this product we see that Euler's constant is deeply related to the gamma function and the poles are clearly the negative or null integers. According to Godefroy [9], Euler's constant plays in the gamma function theory a similar role as π in the circular functions theory.

It's possible to show that Weierstrass form is also valid for complex numbers.

3 Some special values of $\Gamma(x)$

Except for the integer values of $x = n$ for which

$$\Gamma(n) = (n-1)!$$

some non integers values have a closed form.

The change of variable $t = u^2$ gives

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = 2 \int_0^\infty e^{-u^2} du = 2 \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

The functional equation (5) entails for positive integers n (see [1])

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \frac{1.3.5...(2n-1)}{2^n} \sqrt{\pi}, \\ \Gamma\left(n + \frac{1}{3}\right) &= \frac{1.4.7...(3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right), \\ \Gamma\left(n + \frac{1}{4}\right) &= \frac{1.5.9...(4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right), \end{aligned} \tag{10}$$

and for negative values

$$\Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n 2^n}{1.3.5...(2n-1)} \sqrt{\pi}.$$

No basic expression is known for $\Gamma(1/3)$ or $\Gamma(1/4)$, but it was proved that those numbers are transcendental (respectively by Le Lionnais in 1983 and Chudnovsky in 1984).

Up to 50 digits, the numerical values of some of those constants are :

$$\left| \begin{array}{l} \Gamma(1/2) = 1.77245385090551602729816748334114518279754945612238... \\ \Gamma(1/3) = 2.67893853470774763365569294097467764412868937795730... \\ \Gamma(1/4) = 3.62560990822190831193068515586767200299516768288006... \\ \Gamma(1/5) = 4.59084371199880305320475827592915200343410999829340... \end{array} \right. .$$

For example, thanks to the very fast converging formula (which is based on the expression (34) and uses the *Arithmetic-Geometric Mean AGM, [7]*)

$$\Gamma^2(1/4) = \frac{(2\pi)^{3/2}}{AGM(\sqrt{2},1)},$$

this constant was computed to more than 50 millions digits by P. Sebah and M. Tommila [10]. Similar formulae are available for other fractional arguments like $\Gamma(1/3)$ ...

4 Properties of the gamma function

4.1 The complement formula

There is an important identity connecting the gamma function at the complementary values x and $1-x$. One way to obtain it is to start with Weierstrass formula (9) which yields

$$\frac{1}{\Gamma(x)}\frac{1}{\Gamma(-x)} = -x^2 e^{\gamma x} e^{-\gamma x} \prod_{p=1}^{\infty} \left[\left(1+\frac{x}{p}\right) e^{-x/p} \left(1-\frac{x}{p}\right) e^{x/p} \right].$$

But the functional equation gives $\Gamma(-x) = -\Gamma(1-x)/x$ and the equality writes as

$$\frac{1}{\Gamma(x)\Gamma(1-x)} = x \prod_{p=1}^{\infty} \left(1-\frac{x^2}{p^2}\right),$$

and using the well-known infinite product :

$$\sin(\pi x) = \pi x \prod_{p=1}^{\infty} \left(1-\frac{x^2}{p^2}\right)$$

finally gives

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{11}$$

Relation (11) is the *complement (or reflection) formula* and is valid when x and $1-x$ are not negative or null integers and it was discovered by Euler.

For example, if we apply this formula for the values $x = 1/2$, $x = 1/3$, $x = 1/4$ we find

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$
$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi\sqrt{3}}{3},$$
$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}.$$

4.2 Duplication and Multiplication formula

In 1809, Legendre obtained the following *duplication formula* [11].

Theorem 6 *(Legendre, 1809)*

$$\Gamma(x)\Gamma(x+1/2) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x). \tag{12}$$

Proof. Hint : an easy proof can lie on the expression of $\Gamma_p(x)$ and $\Gamma_p(x+1/2)$ from (7), then make the product and find the limit as $p \to \infty$. ■

Notice that by applying the duplication formula for $x = 1/2$, we retrieve the value of $\Gamma(1/2)$, while $x = 1/6$ permits to compute

$$\Gamma\left(\frac{1}{6}\right) = 2^{-1/3}\sqrt{\frac{3}{\pi}}\Gamma^2\left(\frac{1}{3}\right).$$

This theorem is the special case when $n = 2$ of the more general result known as *Gauss multiplication formula* :

Theorem 7 *(Gauss)*

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)...\Gamma\left(x+\frac{n-1}{n}\right) = (2\pi)^{(n-1)/2}n^{1/2-nx}\Gamma(nx)$$

Proof. *Left as exercise.* ■

Corollary 8 *(Euler)*

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)...\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

Proof. *Set $x = 1/n$ in the Gauss multiplication formula.* ■

4.3 Stirling's formula

It's of interest to study how the gamma function behaves when the argument x becomes large. If we restrict the argument x to integral values n, the following result, due to James Stirling (1692-1730) and Abraham de Moivre (1667-1754) is famous and of great importance :

Theorem 9 *(Stirling-De Moivre, 1730) If the integer n tends to infinite we have the asymptotic formula*

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n} n^n e^{-n}. \tag{13}$$

Proof. *See [2] for a complete proof. You may obtain a weaker approximation by observing that the area under the curve $\log(x)$ with $x \in [1, n]$ is well approximated by the trapezoidal rule, therefore*

$$\begin{aligned}\int_1^n \log(x) dx &= n \log(n) - n + 1 \\ &= \sum_{k=1}^{n-1} \frac{\log(k) + \log(k+1)}{2} + R_n = \log((n-1)!) + \frac{1}{2}\log(n) + R_n\end{aligned}$$

and because $R_n = O(1)$ (check this!), we find

$$\log(n!) \approx n \log(n) + \frac{1}{2}\log(n) - n + C$$

which gives this weaker result

$$n! \approx e^C n^n \sqrt{n} e^{-n}.$$

■

Stirling's formula is remarkable because the pure arithmetic factorial function is equivalent to an expression containing important analytic constants like $(\sqrt{2}, \pi, e)$.

There is an elementary way to improve the convergence of Stirling's formula. Suppose you can write

$$n! = \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \ldots\right),$$

then this relation is still valid for $n + 1$

$$(n+1)! = \sqrt{2\pi(n+1)}(n+1)^{(n+1)} e^{-(n+1)} \left(1 + \frac{a_1}{n+1} + \frac{a_2}{(n+1)^2} + \ldots\right) \tag{14}$$

but we also have $(n+1)! = (n+1)n!$ giving

$$(n+1)! = (n+1)\sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \ldots\right). \tag{15}$$

We now compare relations (14) and (15) when n becomes large. This gives after some simplifications and classical series expansions

$$\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+...\right)=\left(1+\frac{1}{n}\right)^{n+1/2}e^{-1}\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+...\right)$$
$$=1+\frac{a_1}{n}+\frac{a_2-a_1+\frac{1}{12}}{n^2}+\frac{\frac{13}{12}a_1-2a_2+a_3-\frac{1}{12}}{n^3}+...$$

and after the identification comes

$$-a_1+\frac{1}{12}=0,$$
$$\frac{13}{12}a_1-2a_2-\frac{1}{12}=0,$$
$$...$$

Therefore we found, by elementary means, the first correcting terms of the formula to be : $a_1=1/12, a_2=1/288, ...$ A more efficient (but less elementary) way to find more terms is to use the *Euler-Maclaurin asymptotic formula.*

In fact the following theorem is a generalization of Stirling's formula valid for any real number x :

Theorem 10 *When $x\to\infty$, we have the famous Stirling's asymptotic formula [1]*

$$\Gamma(x+1)=\sqrt{2\pi x}x^x e^{-x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}-\frac{571}{2488320x^4}...\right). \quad (16)$$

For example here are some approximations of the factorial using different values for n :

n	$n!$	Stirling formula	+ correction $1/(12n)$
5	120	118	119
10	3628800	3598695	3628684
20	2432902008176640000	2422786846761133393	2432881791955971449

5 Series expansion

To estimate the gamma function near a point it's possible to use some series expansions at this point. Before doing this we need to introduce a new function which is related to the derivative of the gamma function.

5.1 The digamma and polygamma functions

Many of the series involving the gamma function and its derivatives may be derived from the Weierstrass formula. By taking the logarithm on both sides of the Weierstrass formula (9) we find the basic relation

$$-\log(\Gamma(x)) = \log(x) + \gamma x + \sum_{p=1}^{\infty}\left(\log\left(1+\frac{x}{p}\right) - \frac{x}{p}\right). \tag{17}$$

5.1.1 Definition

Definition 11 *The psi or digamma function denoted* $\Psi(x)$ *is defined for any non nul or negative integer by the logarithmic derivative of* $\Gamma(x)$*, that is :*

$$\Psi(x) = \frac{d}{dx}\left(\log(\Gamma(x))\right).$$

By differentiating the series (17) we find

$$\begin{aligned}
\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} &= -\gamma - \frac{1}{x} + \sum_{p=1}^{\infty}\left(\frac{1}{p} - \frac{1}{x+p}\right), \\
&= -\gamma + \sum_{p=1}^{\infty}\left(\frac{1}{p} - \frac{1}{x+p-1}\right) \quad x \neq 0, -1, -2, ... \qquad (18) \\
&= -\gamma + \sum_{p=1}^{\infty}\left(\frac{x-1}{p(x+p-1)}\right) \quad x \neq 0, -1, -2, ...
\end{aligned}$$

and those series are slowly converging for any non negative integer x.

5.1.2 Properties

Polygamma functions Now if we go on differentiating relation (18) several times, we find

$$\begin{aligned}
\Psi'(x) &= \frac{\Gamma(x)\Gamma''(x) - \Gamma'^2(x)}{\Gamma^2(x)} = \sum_{p=1}^{\infty}\frac{1}{(p+x-1)^2}, \qquad (19) \\
\Psi''(x) &= -\sum_{p=1}^{\infty}\frac{2}{(p+x-1)^3}, \\
\Psi^{(n)}(x) &= \sum_{p=1}^{\infty}\frac{(-1)^{n+1}n!}{(p+x-1)^{n+1}}, \qquad (20)
\end{aligned}$$

and the $\Psi_n = \Psi^{(n)}$ functions are the *polygamma* functions :

$$\begin{aligned}
\Psi_n(x) &= \frac{d^{n+1}}{dx^{n+1}}\left(\log(\Gamma(x))\right), \\
\Psi_0(x) &= \Psi(x).
\end{aligned}$$

Observe from (19) that for $x > 0$, $\Psi'(x) > 0$ so it's a monotonous function on the positive axis and therefore the function $\log(\Gamma(x))$ is convex when $x > 0$.

Recurrence relations The structure of the series expansion (18) suggests to study

$$\Psi(x+1) - \Psi(x) = \sum_{p=1}^{\infty} \left(\frac{1}{x+p-1} - \frac{1}{x+p} \right)$$

which gives, just like for the gamma function, the recurrence formulae

$$\Psi(x+1) = \Psi(x) + \frac{1}{x},$$
$$\Psi(x+n) = \Psi(x) + \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1} \qquad n \geq 1,$$

and by differentiating the first of those relations we deduce

$$\Psi_n(x+1) = \Psi_n(x) + \frac{(-1)^n n!}{x^{n+1}}. \tag{21}$$

Complement and duplication formulae By logarithmic differentiation of the corresponding complement (11) and duplication (12) formulae for the gamma function we find directly :

Theorem 12

$$\Psi(1-x) = \Psi(x) + \pi \cot \pi x,$$
$$\Psi(2x) = \frac{1}{2}\Psi(x) + \frac{1}{2}\Psi\left(x + \frac{1}{2}\right) + \log(2).$$

5.1.3 Special values of the Ψ_n

Values at integer arguments From the relations (18) and (20) comes

$$\begin{aligned}
\Psi(1) &= -\gamma, \\
\Psi_1(1) &= \zeta(2) = \pi^2/6, \\
\Psi_2(1) &= -2\zeta(3), \\
\Psi_n(1) &= (-1)^{n+1} n! \zeta(n+1),
\end{aligned} \tag{22}$$

where $\zeta(k)$ is the classical *Riemann zeta function.* Using the recurrence relations (21) allow to compute those values for any other positive integer and, for example, we have

$$\begin{aligned}
\Psi(n) = \frac{\Gamma'(n)}{\Gamma(n)} &= -\gamma + \sum_{p=1}^{n-1} \frac{1}{p} \qquad (23) \\
&= -\gamma + H_{n-1}.
\end{aligned}$$

Values at rational arguments The value $\Psi(1/2)$ can be computed directly from (18) or from the psi *duplication formula* with $x = 1/2$:

$$\begin{aligned}\Psi\left(\frac{1}{2}\right) &= -\gamma - 2 + 2\sum_{p=1}^{\infty}\left(\frac{1}{2p} - \frac{1}{2p+1}\right),\\ &= -\gamma - 2 + 2\left(1 - \log(2)\right) = -\gamma - 2\log(2).\end{aligned}$$

To end this section we give the interesting identities

$$\begin{aligned}\Psi\left(\frac{1}{3}\right) &= -\gamma - \frac{3}{2}\log(3) - \frac{\sqrt{3}}{6}\pi,\\ \Psi\left(\frac{1}{4}\right) &= -\gamma - 3\log(2) - \frac{\pi}{2},\\ \Psi\left(\frac{1}{6}\right) &= -\gamma - 2\log(2) - \frac{3}{2}\log(3) - \frac{\sqrt{3}}{2}\pi.\end{aligned}$$

which are consequences of a more general and remarkable result :

Theorem 13 *(Gauss) Let* $0 < p < q$ *being integers*

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log\left(2q\right) + \sum_{k=1}^{q-1}\cos\left(\frac{2\pi kp}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right).$$

Proof. *See [2] for a proof.* ■

From this aesthetic relation, we see that the computation of $\Psi(p/q)$ for any rational argument always involves the three fundamental mathematical constants : $\pi, \gamma, \log(2)$!

5.1.4 Series expansions of the digamma function

The following series expansions are easy consequences of relations (22) and of the series

$$\frac{1}{1+x} - 1 = -\sum_{k=2}^{\infty}(-1)^k x^{k-1}.$$

Theorem 14 *(Digamma series)*

$$\Psi(1+x) = -\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1} \quad |x| < 1, \tag{24}$$

$$\Psi(1+x) = -\frac{1}{1+x} - (\gamma - 1) + \sum_{k=2}^{\infty}(-1)^k\left(\zeta(k) - 1\right)x^{k-1} \quad |x| < 1. \tag{25}$$

5.1.5 Zeros of the digamma function

The zeros of the digamma function are the extrema of the gamma function. From the two relations

$$\begin{aligned} \Psi(1) &= -\gamma < 0 \\ \Psi(2) &= 1 - \gamma > 0, \end{aligned}$$

and because $\Psi'(x) > 0$, we see that the only positive zero x_0 of the digamma function is in $]1, 2[$ and its first 50 digits are :

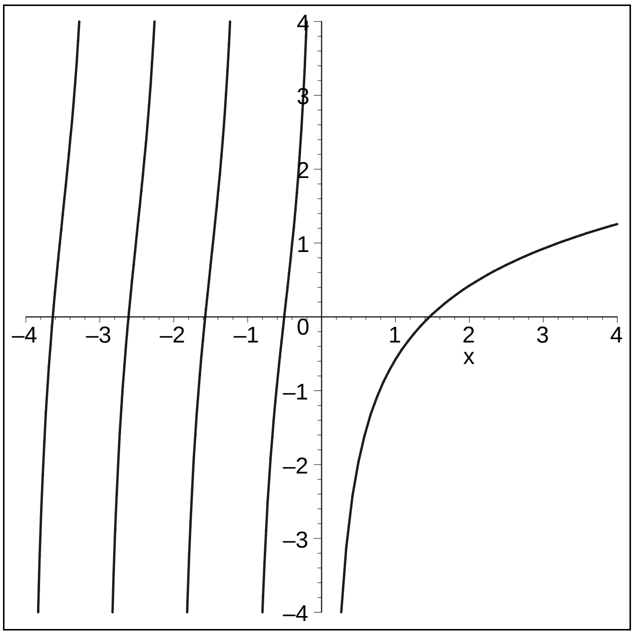
$$\begin{aligned} x_0 &= 1.46163214496836234126265954232572132846819620400644... \\ \Gamma(x_0) &= 0.88560319441088870027881590058258873320795153366990..., \end{aligned}$$

it was first computed by Gauss, Legendre [11] and given in [13]. On the negative axis, the digamma function has a single zero between each consecutive negative integers (the poles of the gamma function), the first one up to 50 decimal places are

$$\left| \begin{aligned} x_1 &= -0.50408300826445540925826930453330249895538518236857 9... \\ x_2 &= -1.57349847316239045877828604369043461265504085911684 6... \\ x_3 &= -2.61072086844414465000153771571872420795107401087348 0... \\ x_4 &= -3.63529336643690109783918156694601771394842386119353 0... \\ x_5 &= -4.65323776174314244171459815114820736371906941613386 8... \end{aligned} \right.$$

and Hermite (1881) observed that when n becomes large [1]

$$x_n = -n + \frac{1}{\log(n)} + o\left(\frac{1}{\log^2(n)}\right).$$



5.2 Series expansion of the gamma function

Finding series expansions for the gamma function is now an easy consequence of the series expansions for the digamma function.

Theorem 15

$$\log(\Gamma(1+x)) = -\gamma x + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} x^k, \quad |x| < 1, \tag{26}$$

$$\log(\Gamma(1+x)) = -\log(1+x) - (\gamma-1)x + \sum_{k=2}^{\infty} \frac{(-1)^k (\zeta(k)-1)}{k} x^k, \quad |x| < 1. \tag{27}$$

Proof. *Use the term by term integration of the Taylor series (24) and (25).*
■

We may observe that the Riemann zeta function at integer values appears in the series expansion of the logarithm of the gamma function. The convergence of the series can be accelerated by computing

$$\frac{1}{2}\left(\log(\Gamma(1+x)) - \log(\Gamma(1-x))\right) = -\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - (\gamma-1)x - \sum_{k=2}^{\infty} \frac{(\zeta(2k+1)-1)}{2k+1} x^{2k+1},$$

We now observe that the complement formula (11) becomes

$$\Gamma(1+x)\Gamma(1-x) = \frac{\pi x}{\sin \pi x}$$

and by taking the logarithms finally

$$\frac{1}{2}\log\left(\Gamma(1+x)\right) + \frac{1}{2}\left(\log\Gamma(1-x)\right) = \frac{1}{2}\log\left(\frac{\pi x}{\sin \pi x}\right)$$

and therefore we obtain the fast converging series due to Legendre :

$$\log\left(\Gamma(1+x)\right) = \frac{1}{2}\log\left(\frac{\pi x}{\sin \pi x}\right) - \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - (\gamma-1)x - \sum_{k=1}^{\infty} \frac{(\zeta(2k+1)-1)}{2k+1} x^{2k+1}, \tag{28}$$

valid for $|x| < 1$.

Gauss urged to his calculating prodigy student Nicolai (1793-1846) to compute tables of $\log(\Gamma(x))$ with twenty decimal places [8]. More modern tables related to $\Gamma(x)$ and $\Psi(x)$ are available in [1].

6 Euler's constant and the gamma function

For $x = 1$ the formula (23) for $\Psi(n)$ yields

$$\Psi(1) = \Gamma'(1) = -\gamma - 1 + H_1 = -\gamma,$$

so Euler's constant is the opposite of the derivative of the gamma function at $x = 1$.

6.1 Euler-Mascheroni Integrals

Using the integral representation of $\Gamma'(x)$ gives the interesting integral formula for Euler's constant

$$\Gamma'(1) = \int_0^\infty e^{-t} \log(t) dt = -\gamma$$

and from (19) comes

$$\Psi'(1)\Gamma^2(1) + {\Gamma'}^2(1) = \Gamma(1)\Gamma''(1)$$

hence

$$\Gamma''(1) = \int_0^\infty e^{-t} \log^2(t) dt = \gamma^2 + \frac{\pi^2}{6}.$$

We may go on like this and compute the *Euler-Mascheroni integrals*

$$\begin{aligned}
\Gamma^{(3)}(1) &= -\gamma^3 - \frac{1}{2}\pi^2\gamma - 2\zeta(3), \\
\Gamma^{(4)}(1) &= \gamma^4 + \pi^2\gamma^2 + 8\zeta(3)\gamma + \frac{3}{20}\pi^4, \\
\Gamma^{(5)}(1) &= -\gamma^5 - \frac{5}{3}\pi^2\gamma^3 - 20\zeta(3)\gamma^2 - \frac{3}{4}\pi^4\gamma - 24\zeta(5) - \frac{10}{3}\zeta(3)\pi^2, \\
&\dots
\end{aligned}$$

6.2 Euler's constant and the zeta function at integer values

Series formulas involving $\zeta(k)$ can also be deduced from formula (26). Taking $x = 1$ gives

$$\log(\Gamma(2)) = -\gamma + \sum_{k=2}^\infty \frac{(-1)^k \zeta(k)}{k},$$

thus

$$\gamma = \sum_{k=2}^\infty \frac{(-1)^k \zeta(k)}{k},$$

which is due to Euler. Setting $x = 1/2$ into (26) gives

$$\log\left(\Gamma\left(\frac{3}{2}\right)\right) = \log(\sqrt{\pi}/2) = -\frac{\gamma}{2} + \sum_{k=2}^\infty \frac{(-1)^k \zeta(k)}{k} \frac{1}{2^k},$$

therefore

$$\gamma = \log\left(\frac{4}{\pi}\right) + 2\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{2^k k}.$$

It is of interest to use the series expansion (28) at $x = 1/2$,

$$\log\left(\Gamma(3/2)\right) = \frac{1}{2}\log\left(\frac{\pi}{2}\right) - \frac{1}{2}\log(3) - \frac{1}{2}(\gamma - 1) - \sum_{k=1}^{\infty}\frac{(\zeta(2k+1)-1)}{2k+1}\frac{1}{2^{2k+1}}.$$

It follows a fast converging expansion for γ

$$\gamma = 1 - \log\left(\frac{3}{2}\right) - \sum_{k=1}^{\infty}\frac{(\zeta(2k+1)-1)}{(2k+1)4^k}.$$

and for large values of k, we have

$$\zeta(2k+1) - 1 = \frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}} + \cdots \sim \frac{1}{2^{2k+1}} \quad \text{hence} \quad \frac{\zeta(2k+1)-1}{4^k} \sim \frac{1}{2}\frac{1}{16^k}.$$

This expression was used by Thomas Stieltjes (1856-1894) in 1887 to compute Euler's constant up to 32 decimal places [14]. In the same article he also computed $\zeta(2)$ up to $\zeta(70)$ with 32 digits.

7 The gamma function and the Riemann Zeta function

The integral definition of the gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt,$$

together with the change of variables $t = ku$ (with k a positive integer) yields

$$\Gamma(x) = \int_0^{\infty}(ku)^{x-1}e^{-ku}k\,du = k^x\int_0^{\infty}u^{x-1}e^{-ku}\,du.$$

We write this in the form

$$\frac{1}{k^x} = \frac{1}{\Gamma(x)}\int_0^{\infty}u^{x-1}e^{-ku}du,$$

hence by summation

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k^x} &= \frac{1}{\Gamma(x)}\int_0^{\infty}u^{x-1}\sum_{k=1}^{\infty}\left(e^{-ku}\right)du \\ &= \frac{1}{\Gamma(x)}\int_0^{\infty}u^{x-1}\left(\frac{1}{1-e^{-u}} - 1\right)du.\end{aligned}$$

We have obtained the beautiful formula

$$\zeta(x)\Gamma(x) = \int_0^\infty \frac{t^{x-1}}{e^t - 1} dt \tag{29}$$

and, for example, for $x = 2$, (29) becomes

$$\frac{\pi^2}{6} = \int_0^\infty \frac{t}{e^t - 1} dt.$$

There is another celebrated and most important functional equation between those two functions, the *Riemann zeta function functional equation* :

Theorem 16 *(Riemann, 1859) Let*

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

an analytic function except at poles 0 *and* 1*, then*

$$\Lambda(s) = \Lambda(1 - s).$$

Proof. *Several proofs may be found in [15]. Euler demonstrated it for integer values of s.* ■

This equation allows to extend the definition of the Zeta function to negative values of the arguments.

8 The Beta function

Let us now consider the useful and related function to the gamma function which occurs in the computation of many definite integrals. It's defined, for $x > 0$ and $y > 0$ by the two equivalent identities :

Definition 17 *The beta function (or Eulerian integral of the first kind) is given by*

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt \tag{30}$$

$$= 2\int_0^{\pi/2} \sin(t)^{2x-1} \cos(t)^{2y-1} dt = 2\int_0^{\pi/2} \sin(t)^{2y-1} \cos(t)^{2x-1} dt, \tag{31}$$

$$= B(y, x).$$

This definition is also valid for complex numbers x and y such as $\Re(x) > 0$ and $\Re(y) > 0$ and Euler gave (30) in 1730. The name *beta function* was introduced for the first time by Jacques Binet (1786-1856) in 1839 [5] and he made various contributions on the subject.

The beta function is symmetric and may be computed by mean of the gamma function thanks to the important property :

Theorem 18 *Let $\Re(x) > 0$ and $\Re(y) > 0$, then*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x). \tag{32}$$

Proof. *We use the definite integral (3) and form the following product*

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty u^{2x-1}e^{-u^2}du\int_0^\infty v^{2y-1}e^{-v^2}dv$$
$$= 4\int_0^\infty\int_0^\infty e^{-\left(u^2+v^2\right)}u^{2x-1}v^{2y-1}dudv,$$

we introduce the polar variables $u = r\cos\theta, v = r\sin\theta$ so that

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta drd\theta$$
$$= 2\int_0^\infty r^{2(x+y)-1}e^{-r^2}dr.2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta d\theta$$
$$= \Gamma(x+y)B(y,x).$$

■

From relation (32) follows

$$B(x+1,y) = \frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)} = \frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} = \frac{x}{x+y}B(x,y),$$

this is the beta function *functional equation*

$$B(x+1,y) = \frac{x}{x+y}B(x,y). \tag{33}$$

8.1 Special values

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi,$$
$$B\left(\frac{1}{3},\frac{2}{3}\right) = \frac{2\sqrt{3}}{3}\pi,$$
$$B\left(\frac{1}{4},\frac{3}{4}\right) = \pi\sqrt{2},$$
$$B(x,1-x) = \frac{\pi}{\sin(\pi x)},$$
$$B(x,1) = \frac{1}{x},$$
$$B(x,n) = \frac{(n-1)!}{x.(x+1)...(x+n-1)} \quad n \geq 1,$$
$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} \quad m \geq 1, n \geq 1.$$

8.2 Wallis's integrals

For example the following integrals (*Wallis's integrals*)

$$W_n = \int_0^{\pi/2} \sin^n \theta d\theta = \int_0^{\pi/2} \cos^n \theta d\theta,$$

may be computed by mean of the beta and gamma functions. Thanks to the relation (31), we have

$$W_n = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right),$$

and come naturally the two cases $n = 2p+1$ and $n = 2p$. For the odd values of the argument n :

$$W_{2p+1} = \frac{1}{2} B\left(p+1, \frac{1}{2}\right) = \frac{\Gamma(p+1)\Gamma(1/2)}{2\Gamma(p+3/2)} = \frac{p!\Gamma(1/2)}{(2p+1)\Gamma(p+1/2)}$$

and using formula (10) produces the well-known result

$$W_{2p+1} = \frac{2^p p!}{1.3.5...(2p+1)} = \frac{4^p p!^2}{(2p+1)!}.$$

The same method permits to compute the integrals for the even values

$$W_{2p} = \frac{1}{2} B\left(p + \frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma(p+1/2)\Gamma(1/2)}{2\Gamma(p+1)}$$

and finally

$$W_{2p} = \frac{1.3.5...(2p-1)}{2^{p+1} p!} \pi = \frac{(2p)!}{4^p p!^2} \frac{\pi}{2}.$$

Observe that it's easy to see that

$$W_{n+2} = \frac{1}{2} B\left(\frac{n+2+1}{2}, \frac{1}{2}\right) = \frac{1}{2} B\left(\frac{n+1}{2} + 1, \frac{1}{2}\right) = \frac{(n+1)/2}{n/2+1} W_n = \left(\frac{n+1}{n+2}\right) W_n$$

thanks to the beta function functional equation (33).

It's interesting to notice that

$$W_\alpha = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)$$

also works for any real number $\alpha > -1$ and therefore we may deduce using (30) and (31) that (respectively with $\alpha = -1/2$ and $\alpha = 1/2$)

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^1 \frac{2dt}{\sqrt{1-t^4}} = \frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}, \tag{34}$$

$$\int_0^{\pi/2} \sqrt{\sin\theta} d\theta = \int_0^1 \frac{2t^2 dt}{\sqrt{1-t^4}} = \frac{(2\pi)^{3/2}}{\Gamma^2(1/4)}.$$

Consequently the product of those two integrals permits to derive the relation due to Euler

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} \int_0^1 \frac{t^2 dt}{\sqrt{1-t^4}} = \frac{\pi}{4}.$$

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